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EDGE-TRANSITIVE LEXICOGRAPHIC AND CARTESIAN PRODUCTS

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Abstract

In this note connected, edge-transitive lexicographic and Cartesian products are characterized. For the lexicographic product $G \circ H$ of a connected graph G that is not complete by a graph H, we show that it is edge-transitive if and only if G is edge-transitive and H is edgeless. If the first factor of $G \circ H$ is non-trivial and complete, then $G \circ H$ is edge-transitive if and only if H is the lexicographic product of a complete graph by an edgeless graph. This fixes an error of Li, Wang, Xu, and Zhao [11]. For the Cartesian product it is shown that every connected Cartesian product of at least two non-trivial factors is edge-transitive if and only if it is the Cartesian power of a connected, edge- and vertex-transitive graph.

Keywords: edge-transitive graph; vertex-transitive graph; lexicographic product of graphs; Cartesian product of graphs.

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1. Introduction

A graph $G = (V(G), E(G))$ is *vertex-transitive* (resp. *edge-transitive*) if the automorphism group $\mathrm{Aut}(G)$ acts transitively on $V(G)$ (resp. $E(G)$). A fine source on the fundamental properties of these graphs, their applications, and related topics is the book [4], the survey [10] and recent papers [3, 12, 20].

While vertex-transitivity of graph products is well understood, cf. [6], it is rather surprising that not much can be found in the literature about their edge-transitivity. It is claimed in [11] that the lexicographic product of edge-transitive graphs is edge-transitive as well. This is not true, as we shall show in the next section, in which we will characterize edge-transitive lexicographic products. This will be done in two steps, first for products whose first factors are not complete (Theorem 3) and then for products whose first factors are complete (Theorem 4). In Section 3, we characterize connected, edge-transitive Cartesian products.

If G is a graph, then its connectivity is denoted by $\kappa(G)$. Recall that $\kappa(G) \leq \delta(G)$, where $\delta(G)$ is the minimum degree in G. We will denote the edgeless graph on m vertices by N_m.

The *lexicographic product* $G \circ H$ of graphs G and H is the graph with $V(G \circ H) = V(G) \times V(H)$, where (g, h) is adjacent to (g', h') if either $g = g'$ and $hh' \in E(H)$, or $gg' \in E(G)$.

The *Cartesian product* $G \,\square\, H$ of graphs G and H is the graph with the vertex set $V(G) \times V(H)$, vertices (g, h) and (g', h') being adjacent if $g = g'$ and $hh' \in E(H)$, or $gg' \in E(G)$ and $h = h'$. If $u = (g, h) \in V(G \,\square\, H)$, then the subgraph of $G \,\square\, H$ induced by the vertices of the form (g, x), $x \in V(H)$, is isomorphic to H; it is denoted with H^u and called the *H-layer* (through u). Analogously G-layers are defined. The same terminology applies to lexicographic products.

Vertices x and y of a graph G are in *relation R_G* (or simply in *relation R* if the graph G is clear from the context) if they have the same open neighborhood, that is, if $N_G(x) = N_G(y)$ holds. It is well-known (cf. [6, Exercise 8.4]) that R is an equivalence relation on $V(G)$; its equivalence classes are called *R-classes.* Graphs in this paper have no loops, hence no two vertices of an R-class are adjacent. Finally, we introduce the *relation S_G* on $V(G)$ by letting x, y be in relation S_G if they have the same closed neighborhoods. Again, S_G is an equivalence relation, and its equivalence classes are called *S-classes.*

2. Edge-transitive lexicographic products

The lexicographic product of graphs is also known as the composition of graphs as well as graph substitution. The latter is due to the fact that $G \circ H$ can be obtained from G by substituting a copy H_g of H for every vertex g of G, and then joining all vertices of H_g with all vertices of $H_{g'}$ if $gg' \in E(G)$. This graph operation was of ongoing interest in the last several decades, the references [1, 5, 9] present just a very selective list of recent papers.

It is well-known that a lexicographic product $G \circ H$ is vertex-transitive if and only if G and H are vertex-transitive, see [6, Theorem 10.14]. In [11, Theorem 2.2] it is claimed that if G and H are edge-transitive graphs, then $G \circ H$ is edge-transitive as well. To see that this need not be the case, consider the edge-transitive graphs K_2 and P_3 and their lexicographic product $K_2 \circ P_3$, which is clearly not edge-transitive. More generally, the following holds.

Proposition 1. *Suppose that G, H, and $G \circ H$ are edge-transitive and that each of G and H has at least one edge. Then H is vertex-transitive.*

Proof. Suppose on the contrary that H is not vertex-transitive. Then, by the above, $G \circ H$ is not vertex-transitive. Because both G and H have at least one edge, $G \circ H$ is not bipartite. Since an edge-transitive graph that is not vertex-transitive is necessarily bipartite (cf. [19, Proposition 2.2] or [4, Lemma 3.2.1]), it follows that $G \circ H$ is not edge-transitive, a contradiction. ■

To characterize edge-transitive lexicographic products whose first factor is not complete, we will make use of the following result due to Watkins.

Theorem 2. [18, Corollary 1A] *If G is a connected, edge-transitive graph, then $\kappa(G) = \delta(G)$.*

Our first result now reads as follows.

Theorem 3. *Let G be a connected graph that is not complete and H be any graph. Then $G \circ H$ is edge-transitive if and only if G is edge-transitive and H is edgeless.*

Proof. Suppose first that G and H are as stated and that $G \circ H$ is edge-transitive. From [6, Proposition 25.7] we know that, as G is not complete, $\kappa(G \circ H) = \kappa(G)\,|V(H)|$. Since $\delta(G \circ H) = \delta(H) + \delta(G)|V(H)|$, Theorem 2 implies that

$$\kappa(G)\,|V(H)| = \delta(H) + \delta(G)|V(H)|\,.$$

As $\kappa(G) \le \delta(G)$ holds, we infer that $\delta(H) = 0$ (and that $\kappa(G) = \delta(G)$). Hence H is edgeless.

It is easy to see that the R-classes of $A = G \circ H$ are the products of the R-classes of G with H. The automorphisms of A preserve the R-classes, hence every automorphism of A induces an automorphism of A/R. Since $\mathrm{Aut}(A)$ is edge-transitive, this is also the case for A/R. Notice that $A/R = A/R_{G\circ H}$ is the image of the projection of $(G/R_G) \circ H$ onto G/R_G, hence $A/R_{G\circ H} \cong G/R_G$. Therefore G/R_G is edge- transitive, and thus so is G.

The converse is straightforward. ■

Interestingly, we did not need to know much about the automorphism group of the lexicographic product for the proof of this theorem. However, for the next theorem, which characterizes the edge-transitive lexicographic products whose first factors are complete, we need more information.

Given a lexicographic product $G \circ H$, a vertex $(g,h) \in V(G \circ H)$ and a $\beta \in \mathrm{Aut}(H)$, then the permutation of $V(G \circ H)$ that maps (g,h) into $(g,\beta h)$ for every $h \in V(H)$, and fixes all other vertices of $G \circ H$, clearly is an automorphism of $G \circ H$. Furthermore, if $\alpha \in \mathrm{Aut}(G)$, then the mapping $(g,h) \mapsto (\alpha g, h)$ defined on $V(G \circ H)$ also is in $\mathrm{Aut}(G \circ H)$.

The group generated by such elements is the *wreath product* of $\mathrm{Aut}(G)$ by $\mathrm{Aut}(H)$ and denoted $\mathrm{Aut}(G) \circ \mathrm{Aut}(H)$. Clearly it is a subgroup of $\mathrm{Aut}(G \circ H)$, and all elements of $\mathrm{Aut}(G) \circ \mathrm{Aut}(H)$ can be written in the form

$$(g,h) \mapsto (\alpha g, \beta_g h),$$

where $\alpha \in \mathrm{Aut}(G)$ and every β_g is in $\mathrm{Aut}(H)$. Notice that two vertices that have the same G-coordinate, say (g,h) and (g,h'), are mapped into vertices that have the same G-coordinate again, namely into $(\alpha g, \alpha_g h)$ and $(\alpha g, \alpha_g h')$. Evidently this means that $\mathrm{Aut}(G) \circ \mathrm{Aut}(H)$ preserves H-layers.

By Sabidussi [15], a necessary and sufficient condition that $\mathrm{Aut}(G \circ H) = \mathrm{Aut}(G) \circ \mathrm{Aut}(H)$ is that H be connected if R_G be non-trivial, and that the complement $\overline{H}$ of H be connected if S_G be non-trivial.

Theorem 4. *The lexicographic product $G \circ H$ of a non-trivial complete graph G by a graph H is edge-transitive if and only if H is the product of a complete graph by an edgeless graph. This means that $G \circ H$ can be represented in the form $K \circ N$, where K is complete and N edgeless.*

Proof. Let $A = G \circ H$. Clearly A is connected and has at least two H-layers.

We first treat the case where $\mathrm{Aut}(A) = \mathrm{Aut}(G) \circ \mathrm{Aut}(H)$. Suppose H contains an edge. Then all H-layers contain an edge and, since $\mathrm{Aut}(A)$ preserves the H-layers, all edges are in H-layers by edge-transitivity. But then A is disconnected. Hence H is edgeless, and $A = K \circ N$ for $K = G$ and $N = H$.

Now, let us assume that $\mathrm{Aut}(A) \neq \mathrm{Aut}(G) \circ \mathrm{Aut}(H)$. Since G is a non-trivial complete graph, S_G is non-trivial, and therefore $\overline{H}$ must be disconnected by Sabidussi's theorem. Let $B_1, \ldots, B_\ell$ be the complements of the connected components of $\overline{H}$, that is,

$$\overline{H} = \overline{B}_1 \cup \cdots \cup \overline{B}_\ell \,.$$

H is then the *join* of the B_i, $1 \le i \le \ell$, that is, H consists of the B_i and every vertex in B_i is joined by an edge to every vertex in B_j for $j \neq i$. The B_i are the *join components* of H. We will use the notation B_i^v for the subgraph of A that is induced by the vertices $\{(v,x) \,|\, x \in V(B_i)\}$. It is isomorphic to B_i.

Suppose B_j contains an edge. Since every B_j^v contains an edge, let $e \in E(B_j^v)$. By edge-transitivity e must be mapped by some automorphism α to an edge αe whose endpoints are in different H-layers. But then, $\alpha(B_j^v)$ has a disconnected complement, and hence so does B_j, contrary to the definition of the B_i as the connected components of $\overline{H}$. Thus all B_i are edgeless.

Suppose that some B_1 and B_2 have different numbers of vertices. Then any two vertices $v \in V(B_1)$ and $w \in V(B_2)$ have different degrees in H. For any two vertices $x, y \in G$ the edge $[(x,v),(y,v)]$ of $G \circ H$ will have endpoints of degree $d_H(v)+(n-1)|V(H)|$, whereas the degrees of the endpoints of $[(x,w),(y,w)]$ are $d_H(w)+(n-1)|V(H)|$. Since $d_H(v) \neq d_H(w)$ these edges cannot be mapped into each other. Hence all B_i are edgeless graphs with the same number of vertices, say r. Then $H = K_{|V(H)|/r} \circ N_r$. Using the fact that the lexicographic product is associative we conclude that

$$A = K_n \circ \big(K_{|V(H)|/r} \circ N_r\big) = \big(K_n \circ (K_{|V(H)|/r}\big) \circ N_r = K \circ N_r\,,$$

where $K = K_n \circ K_{|V(H)|/r}$. ■

Consider the following illustrative example to Theorem 4. It is easy to see that the lexicographic product $A_n = K_n \circ C_4$ is the so-called cocktail-party graph of order $4n$. Clearly, A_n is edge-transitive. Now, A_n can also be represented as $A_n = K_{2n} \circ N_2$, in accordance with Theorem 4.

A closely related result should be mentioned here. Recall that a graph is *super-connected* if every minimum vertex cut isolates a vertex. Then Meng [13] proved that a connected, vertex- and edge-transitive graph G is not super-connected if and only if G is isomorphic to $C_n \circ N_m$, $(n \geq 6, m \geq 1)$, or to $L(Q_3) \circ N_m$ $(m \geq 1)$, where $L(Q_3)$ is the line graph of the 3-cube.

3. Edge-transitive Cartesian products

It is well-known that a Cartesian product of connected graphs has transitive automorphism group if and only if every factor has transitive automorphism group, see [6, Proposition 6.16]. On the other hand, vertex- and edge-transitivity of the factors does not imply in general that their Cartesian product is edge-transitive. A simple example is $K_3 \,\square\, K_2$, and, more generally, $K_n \,\square\, K_m$, where $n, m \geq 2$ and $n \neq m$. Indeed, since K_n and K_m are relatively prime, no automorphism of $K_n \,\square\, K_m$ maps an edge of a K_n-layer onto an edge of a K_m-layer.

The main result of this section is the characterization of edge-transitive connected Cartesian products. For the proof we will use the structure of the automorphism group of Cartesian products of connected prime graphs and the result of Sabidussi [16] and Vizing [17] that every connected graph G has a unique prime factor representation with respect to the Cartesian product.

To be more precise, every connected graph G can be represented as a product $H_1 \,\square\, \cdots \,\square\, H_k$ of connected, prime graphs, and the presentation is unique up to the order and isomorphisms of the factors. It is convenient to denote the vertices x of G as vectors $(x_1, \ldots, x_k)$, where $x_i \in V(H_i)$, $1 \leq i \leq k$. Then every $\varphi \in \mathrm{Aut}(G)$ can be represented in the form

$$\varphi(x)_i = \varphi_i(x_{\pi(i)}), \tag{1}$$

where $1 \leq i \leq k$, $\varphi_i \in \mathrm{Aut}(H_i)$, and π is a permutation of the set $\{1, \ldots, k\}$. This result is due to Imrich and Miller [7, 14]; see also [6, Theorem 6.10]. There are two important special cases.

In the first case π is the identity permutation and only one φ_i is nontrivial. Then the mapping φ_i^* defined by

$$\varphi_i^*(x_1, \ldots, x_k) = (x_1, \ldots x_{i-1}, \varphi_i(x_i), x_{i+1}, \ldots, x_k)$$

is an automorphism and we say that φ_i^* is *induced by the automorphism* φ_i *of the factor* H_i. Note that φ_i^* preserves every H_i-layer and preserves every set of H_j-layers for fixed j.

The second case is the transposition of isomorphic factors, which is possible if G has two isomorphic factors, say $H_i \cong H_j$. To simplify notation we can assume that $H_i = H_j$, where $i < j$. Then the mapping $\varphi_{i,j}$ defined by

$$\varphi_{i,j}(x_1, \ldots, x_i, \ldots, x_j, \ldots, x_k) = (x_1, \ldots, x_j, \ldots, x_i, \ldots, x_k)$$

is an isomorphism. We call it a *transposition of isomorphic factors*. Clearly $\varphi_{i,j}$ interchanges the set of H_i-layers with the set of H_j-layers.

It is easily seen that the automorphisms that are induced by the automorphisms of the factors, together with the transposition of isomorphic factors generate $\mathrm{Aut}(G)$. Hence every automorphism φ of G permutes the sets of H_i-layers in the sense that φ maps the set of H_i-layers into the set of $H_{\pi(i)}$-layers, where π is the permutation from Equation (1).

As an easy application of the above we determine the number of vertex orbits of powers of prime, connected graphs with two vertex orbits.

Lemma 5. *Let H be a connected graph with two vertex orbits that is prime with respect to the Cartesian product, and let k a positive integer. Then H^k has $k+1$ vertex orbits.*

Proof. Let the vertex orbits of H be V_0 and V_1. Then $V_0 \cup V_1 = V(H)$ and if $x \in V(H^k)$, then every component x_i of x can be in V_0 or V_1. Let X_r be the set of vertices where r components are in V_0. Clearly $r \in \{0, 1, \ldots, k\}$, hence there are $k+1$ such sets. Furthermore, every automorphism of H^k that is induced by one of the factors preserves all X_r, and the same is true for transpositions of isomorphic factors. Since $\mathrm{Aut}(H^k)$ is generated by these automorphisms, the X_r are preserved by $\mathrm{Aut}(H^k)$.

The observation that $\mathrm{Aut}(H^k)$ acts transitively on every X_r completes the proof. ■

Theorem 6. *A connected graph that is not prime with respect to the Cartesian product is edge-transitive if and only if it is the power of a connected, edge- and vertex-transitive graph.*

Proof. Suppose $H_1 \,\square \cdots \square\, H_k$ is the prime factorization of an edge-transitive graph G. Let e be an edge in an H_1-layer of G and f be an edge in an H_i-layer, $1 \leq i \leq k$. By edge-transitivity there is an automorphism φ that maps e into f. Clearly φ maps the H_1-layer containing e into the H_i-layer containing f. Hence all factors are isomorphic. If f is in the same H_1-layer as e, then φ maps this H_1-layer, say H_1^v, into itself. Thus $\varphi|H_1^v$ is an automorphism of H_1^v. Since f was arbitrarily chosen, the action of the restriction of $\varphi|H_1^v$ on H_1^v is edge-transitive. As $H_1 \cong H_1^v$ it is clear that $G \cong H^k$ for some graph $H \cong H_1$.

If H is not vertex-transitive, then G has at least three vertex orbits under the action of $\mathrm{Aut}(G)$ by Lemma 5. However, an edge-transitive graph can have only one or two.

To complete the proof we have to show that every connected graph G is edge-transitive if it is a product of the form $H_1 \,\square \cdots \square\, H_k$, where the factors are isomorphic copies of an edge- and vertex-transitive graph H.

For, given edges $e \in H_i^v$ and $f \in H_j^w$, where $w = (w_1, \ldots, w_k)$, we first apply $\pi_{i,j}$, the automorphism that interchanges the i-th with the j-th coordinate of every vertex $v \in V(G)$. Hence $\pi_{i,j}(e) \in H_j^{\pi_{i,j}(v)}$. Then we choose automorphisms $\alpha_i \in \mathrm{Aut}(H_i)$ for every $i \in \{1, \ldots, k\}$ such that $\alpha_i(\pi_{i,j}(v)_i) = w_i$. Setting $\alpha = (\alpha_1, \ldots, \alpha_k)$, we thus have $w = (\alpha\pi_{i,j})(v)$, and $\alpha\pi_{i,j}(e) \in H_j^w$. Since H_j is edge-transitive, there clearly exists an automorphism φ of G that maps $\alpha\pi_{i,j}(e)$ into f. Hence $f = \varphi\alpha\pi_{i,j}(e)$. ■

We comment here that the misstatement in [21] (a Cartesian product is edge transitive if and only if its factors are) should read simply "only if" and not "if and only if".

There are graphs that are vertex- and edge-transitive, but, given any edge e, there is no automorphism that interchanges the endpoints of e. Such graphs are called *half-transitive*, cf. [2]. By Theorem 6 any connected half-transitive graph G is either prime or the Cartesian power of a prime, vertex- and edge-transitive graph H. It is easy to see that H must be also be half-transitive, and that any Cartesian power of a half-transitive graph is also half-transitive. We thus have the following corollary:

Corollary 7. *A connected graph that is not prime with respect to the Cartesian product G is half-transitive if and only if it is the power of a connected, half-transitive graph.*

We conclude with the remark that so-called weak Cartesian products, that is, connected components of Cartesian products with infinitely many factors, can be vertex-transitive, even if no factor is vertex-transitive, see [8].

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